An Improved Van Driest Skin-Friction Formula for Compressible Turbulent Boundary Layers

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The Van Driest transformation, which is an exact consequence of innerlayer similarity, has frequently been used successfully to correlate velocity profiles throughout the boundary layer. Here we deduce the corresponding skin-friction formula, provisionally named "Van Driest III," which is similar to that of Fernholz but in a more convenient form for making use of new profile data.

Nomenclature

a = speed of sound
B_q = heat-transfer parameter, Eq. (8e)
C, C_l = constants of integration, Eqs. (7) and (8c)
c_f = $\tau_w/(0.5\rho_e U_e^2)$, skin-friction coefficient
c_p = specific heat at constant pressure
K, K_θ = mixing length constants, Eqs. (1) and (2)
M_τ = u_τ/a_w "friction Mach number," Eq. (8d)
Pr = Prandtl number
Q = heat-transfer rate from wall to fluid
r = recovery factor, 0.89 approximately
T = temperature
U = x-component velocity
U^* = transformed (low-speed) U, Eqs. (8c) and (9)
u_τ = $(\tau_w/\rho_w)^{1/2}$, friction velocity
x, y = coordinates along and normal to wall
γ = ratio of specific heats
δ = boundary-layer thickness [free parameter in fitting Eq. (9)]
θ = momentum thickness
Π = wake parameter, Eq. (9)
υ = kinematic viscosity
ρ = density
τ = total (viscous plus turbulent) shear stress

Subscripts

e = external flow value
w = wall value

Introduction

THE Van Driest transformation, as given in its most general form by Rotta,[1] uses the "mixing length" formulas for velocity and temperature (which are exact consequences of inner-layer similarity) together with the mean-momentum and mean-enthalpy equation with the convective terms and pressure gradient neglected. Respectively, these are

$$\partial U/\partial y = (\tau/\rho)^{1/2}/Ky \tag{1}$$

$$\partial T/\partial y = -Q/[\rho c_p (\tau/\rho)^{1/2} K_\theta y] \tag{2}$$

$$\partial \tau/\partial y = 0 \tag{3}$$

$$\partial Q/\partial y = \tau \partial U/\partial y \tag{4}$$

Here $K(\approx 0.41)$ and $K_\theta(\approx 0.45)$ are absolute constants, K/K_θ being the turbulent Prandtl number Pr_t. From Eqs. (1) and (2)

$$(Q/c_p)\,\partial U/\partial y = -(K_\theta/K)\tau \partial T/\partial y \tag{5}$$

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Substituting for Q and T from the trivial integrals of Eqs. (3) and (4), and integrating, gives

$$T = C_l T_w - (K/K_\theta)[Q_w U/(c_p \tau_w) - U^2/(2c_p)] \tag{6}$$

The constant of integration C_l is not necessarily 1.0, because Eqs. (1) and (2) do not apply in the viscous sublayer, where

$$(\tau_w/\rho)^{1/2} y/\upsilon < 40$$

(say) or the conductive sublayer where $[(\tau_w/\rho)^{1/2} y/\upsilon]Pr < 40$. Taking $\rho/\rho_w = T_w/T$ to give ρ in Eq. (1) from Eq. (6), and writing $u_\tau \equiv (\tau_w/\rho_w)^{1/2}$, integration of Eq. (1) gives

$$U/u_\tau = (C_l^{1/2}/R)\sin(RU^*/u_\tau) - H[1-\cos(RU^*/u_\tau)] \tag{7}$$

where

$$R = [0.5(\gamma-1)K/K_\theta]^{1/2} M_\tau \tag{8a}$$

$$H = Q_w/(\rho_w u_\tau^3) \equiv B_q/[(\gamma-1)M_\tau^2] \tag{8b}$$

$$U^*/u_\tau = (1/K)\ln(u_\tau y/\upsilon_w) + C \tag{8c}$$

and M_τ and B_q, defined by

$$M_\tau = u_\tau/[(\gamma-1)c_p T_w]^{1/2} \equiv u_\tau/a_w \tag{8d}$$

$$B_q = Q_w/(\rho_w c_p u_\tau T_w) \tag{8e}$$

are the proper inner layer Mach number and heat-transfer parameter. (Rotta's β_q is $-B_q$: the K_q of Meier[2] is $(KK_\theta)^{1/2}$). In general, the constants of integration C and C_l are functions of M_τ and B_q and possibly of gas properties, and, at best, the Van Driest transformation in Eq. (7) applies rigorously only in the region outside the sublayers but approximately inside the line $y/\delta = 0.2$.

Rotta suggested that the complete velocity profile, outside the sublayers but inside the line $y/\delta = 1$, could be represented empirically by adding a "wake" contribution $(\Pi/K)w(y/\delta)$ to Eq. (7), analogous to the highly-successful low-speed profile family of Coles. Taking $w(1)=2$ so that w can be approximated by $1\text{-}\cos(\pi y/\delta)$ or a similar analytic curve. $\Pi \approx 0.58$ in a constant-pressure low-speed boundary layer with $U_e\theta/\upsilon_e > 5000$ but decreases at lower Reynolds number. Several later workers have preferred to add the wake contribution to Eq. (8c) which is then exactly Coles' low-speed profile

$$U^*/u_\tau = (1/K)[\ln(u_\tau y/\upsilon_w) + KC + \Pi(1-\cos \pi y/\delta)] \tag{9}$$

and Eq. (7) with the substitution Eq. (9) is then found to be a good fit to measured profiles in high-speed flow. The most general form of the Van Driest skin-friction formula as expounded by Rotta[1] follows by writing Eq. (7) with Eq. (9) at $y=\delta$ where $U=U_e$. As normally used, however, the Van Driest formulas implicitly take $C_t=1$, $K=K_\theta$ and assume that U_e^*/u_τ can be obtained from any low-speed skin-friction law at an empirically-defined reference Reynolds number. Strictly, the Reynolds number common to the high-speed flow and its low-speed transformation is $u_\tau\delta/\upsilon_w$, as can be seen from Eq. (8c), but in Van Driest I, the chosen reference was $(\rho_w \rho_e)^{1/2}U_e\theta/\mu_w$, changes in Van Driest II to $\rho_e U_e\theta/\mu_w$. Thus, these formulas can make no use of velocity profile measurements and can be improved only by changing the choice of reference Reynolds number. As shown by Hopkins and Inouye[3] the results on highly-cooled walls are rather poor. In the present paper we describe the general version, deriving most of the empirical information directly from the consensus of velocity-profile measurements but adjusting the constant C to optimize agreement with the c_f data in Ref. 3.

Although several workers have made some use of the general Van Driest transformation to derive c_f values from Eq. (9), it does not seem to have been used to derive a complete c_f law making use of experimental profile data.

The Skin-Friction Formula

The Van Driest skin-friction formulas were intended for hand calculation: for what is nowadays a negligible extra completion we can use Eq. (7) directly. Given M_e, $U_e\theta/\upsilon_e$ and T_w/T_e, we guess c_f and θ/δ and deduce $u_\tau\delta/\upsilon_w$. We then use Eq. (9) in Eq. (7) to get a new value of c_f (setting $y=\delta$) and tabulate U/U_e against y/δ. We thence evaluate θ/δ, using Eq. (6) to give ρ, deduce a new value of $u_\tau\delta/\upsilon_w$, and iterate to convergence. The contribution to θ from the viscous sublayer is estimated in the current program by assuming $U\propto y^{1/6}$ between $y=0$ and $y=0.05\delta$, with an average value of temperature. Using the Sommer-Short intermediate-temperature formula[3] to give initial guesses, convergence of c_f to 0.1% accuracy takes less than ten iterations, a fraction of a second on a modern computer: the program is given in Ref. 4. Values of C and C_t as functions of M_τ and B_q, and a value of Π as a function of M_τ, B_q, and an appropriate Reynolds number, must be specified.

Most of the successful applications of the Van Driest transformation to high-speed velocity profiles have used. Eq. (6), or the closely-equivalent Crocco quadratic temperature profile, to reduce the data, although it is known (e.g., Ref. 2) that this profile is not very accurate in the outer layer of boundary layers on cold walls. However, the only effect of departures from the assumed temperature variation Eq. (6) is to alter the value of Π that gives the best fit to the profile. If the departures are uniquely related to the Mach number, heat-transfer parameter and Reynolds number, one simply obtains a different Π correlation. A practical difficulty in the use of any "flat-plate" formulas is the effect of nonuniform wall temperature upstream of the measurement point, which can cause large distortions of the temperature profile.

Velocity profile data accompanied by direct measurements of skin friction suggest that C increases with M_τ, while the very large overestimates of c_f on very cold walls ($T_w/T_{aw}<0.1$) demonstrated by Hopkins and Inouye suggest that C increases, or C_t decreases, if B_q is negative (cold wall). Temperature profile measurements are scattered, and show no definite evidence that C_t differs from unity. Taking $C_t=1$ in Eq. (6) and Eq. (7), the values of C required in Eq. (9) to give agreement to about 5% with the c_f results in air quoted by Hopkins and Inouye, with $K/K_\theta=0.91$ and $\Pi=0.58$ (ignoring low Reynolds number effects), are well fitted by

$$C=5.2+95M_\tau^2+30.7B_q+226B_q^2 \qquad (10)$$

As shown in Ref. 5, the value of C required to fit constant-density data, with c_f deduced from Preston tube readings, is 5.2 rather than the value of 5.0 usually quoted.

Fernholz[6] has correlated a wake parameter similar to Π as a function of $\rho_e U_e\theta/\mu_w$ only, on adiabatic walls with $M_e<5$; the high-Reynolds-number value of Π is close to the low-speed value of 0.58. Elfstrom[7] has found good agreement, on a *cold* wall at $M_e\sim 9$, with the low-speed correlation for Π as a function of $u_\tau\delta/\upsilon_w$. These results suggest that Π does not depend directly on M_τ or B_q, and we therefore take $\Pi=0.58$ at high Reynolds numbers. The allowance for variations of Π at low Reynolds number is discussed in the following.

The object of using this more complicated version of the Van Driest skin-friction formula, for which we suggest the name "Van Driest III," is to permit logical improvement by use of velocity profile measurements to define the behavior of C, C_t and Π. Fernholz's skin-friction law[6] is conceptually similar to the present one, but difficult to relate directly to profile measurements. Existing skin-friction formulas like those reviewed by Hopkins and Inouye can be improved only by arbitrarily adjusting their transformation factors to yield a better fit to c_f data. The perils of this are well demonstrated by the Spalding-Chi formula, which is effectively identical to the Van Driest II formula in adiabatic flows but uses a function of T_w/T_{aw} in the Reynolds-number transformation factor. This gives improved agreement near $T_w/T_{aw}=0.2$ but poorer agreement between 0.2 and 1.0. Moreover, most of the existing formulas have transformation factors which are independent of Reynolds number, and are therefore unlikely to represent low-Reynolds-number effects on the outer layer correctly. A skin-friction formula explicitly related to velocity-profile behavior, as in the highly successful low-speed c_f formula based on Eq. (9), allows us to unify profile measurements and c_f measurements from different experiments, and can also be used to correlate c_f in arbitrary pressure gradients in terms of integral parameters of the velocity profile.

The Scaling of Low-Reynolds-Number Effects

The work of Fernholz[6] implies that the velocity profile parameter Π correlates as a function of $\rho_e U_e\theta/\mu_w$ on adiabatic walls for Mach number up to 5 at least. Since the viscosity at the wall is most unlikely to affect the outer layer directly it is best to interpret Fernholz's Reynolds number as $(1+0.5\ r(\gamma-1)M_e^2)^{0.76}U_e\theta/\nu_e$ taking the exponent of the power law for viscosity as a function of temperature as 0.76 at typical *wall* temperatures.

Now Morkovin's hypothesis[8] suggests that fluctuations of pressure or density should not affect boundary layers at $M_e<5$. The quantity most likely to affect the behavior of the large eddies or the irrotational/turbulent interface is the density gradient in the outer layer. A suitable parameter for correlating this is $\rho_{0.2}/\rho_e$ or $T_{0.2}/T_e$ where suffix 0.2 denotes conditions at $y/\delta=0.2$, the nominal inner edge of the outer layer. This parameter is still suitable if the real cause of Mach-number variation of low-Reynolds-number effects is viscosity

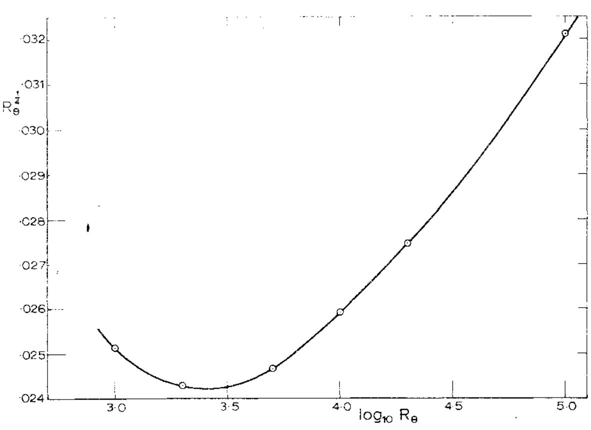

Fig. 1 The incompressible skin-friction law of Coles.

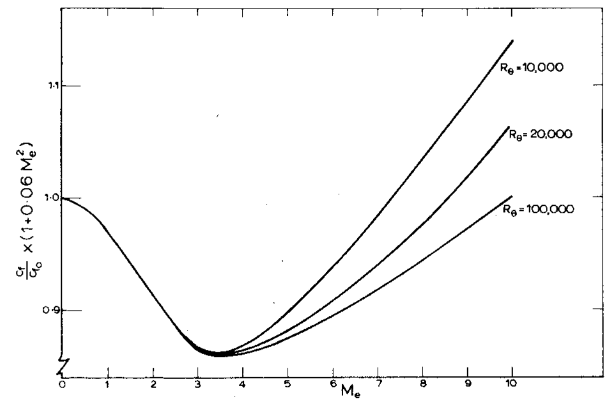

Fig. 2 Adiabatic-wall skin-friction ratio.

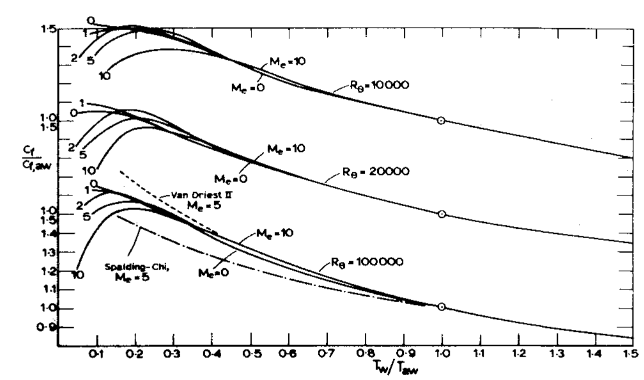

Fig. 3 Ratio of c_f to adiabatic-wall value for three different Reynolds numbers. Note scale displacement.

fluctuations near the interface, because these will be fairly closely related to the mean temperature gradient.

The right Reynolds number to use in correlating viscous effects at the interface is $(\tau_w/\rho_e)^{1/2}\delta/\nu_e \equiv R_\nu$: turbulence intensities correlate quite well as $\rho\overline{u^2}/\tau_w = f(y/\delta)$, so $(\overline{u^2})^{1/2} \propto (\tau_w/\rho)^{1/2}$ at given y/δ. Therefore, low Reynolds-number effects should correlate with $R_\nu f_1(T_{0.2}/T_e)$. Obtaining the function f_1 by forcing agreement with Fernholz's result that the onset of low-Reynolds number effects in adiabatic boundary layers occurs at the low-speed value of $\rho_e U_e \theta/\mu_w$, about 5000, we can deduce the value of $U_e\theta/\nu_e \equiv R_\theta$ below which the effects begin, as a function of M_e and T_w/T_{aw}. We obtain $T_{0.2}/T_e$ by using the Crocco formula with a one-sixth power law velocity profile, appropriate to fairly low Reynolds numbers. Auxiliary relations for θ/δ and c_f are needed: for the former we have used the simple formula given by Stollery[9] and for the latter an analytic approximation to the "Van Driest III" values. The results, which need be – and are – only rough ones, can be represented for $M_e < 10$, $T_w/T_{aw} > 0.1$ by

$$R_{\theta,\min} = 5000(1+0.1M_e^2)[1-0.3(1.0-T_w/T_{aw})] \tag{11}$$

where the Mach-number factor is an adequate approximation to

$$(1+0.178M_e^2)^{0.76}$$

The results are affected only slightly if $T_{0.4}$ is used instead of $T_{0.2}$: the attribution of compressibility effects on $R_{\nu,\min}$ to the density gradient in the outer layer is speculative (though plausible) but at least the choice of density-gradient parameter is not critical. Strictly Π should be correlated using τ_w/ρ_e and δ as scales but for practical purposes correlation on $R_\theta/R_{\theta,\min}$ should be adequate.

Therefore, we suggest the use of

$$\Pi = f_2(R_\theta/R_{\theta,\min}) \tag{12}$$

where $R_{\theta,\min}$ is given by Eq. (11) and an adequate analytic fit to low-speed data for the function f is

$$f_2(x) = 0.58(1-1.42\exp(-4.35x)) \tag{13}$$

This allowance for low-Reynolds-number effects is incorporated in the program.

Charts

Charts for use at high Reynolds number are given in Figs. 1 to 3. At Reynolds numbers below about $R_\theta = 10^4$, depending on Mach number and temperature ratio, c_f starts to change rather rapidly with R_θ because of the variation of Π. Charts for these low R_θ values would be complicated, and are omitted partly for this reason and partly because the Π variation is poorly documented. Values for low Reynolds numbers can be obtained directly from the computer program. The data used to prepare the charts, especially the correlation Eq. (10) for C, come from high-speed airflows and although the formulas all apply to low-speed flows with large temperature differences there are not enough results available to check the predicted variation of c_f with T_w/T_{aw} at low M_e. The decrease in C_f on very cold walls at high M_e is not predicted by other methods but follows from the data analysis of Hopkins and Inouye. The scaling functions in the ordinates of Figs. 1 and 2 were chosen to minimize variations of the ordinate and have no physical significance.

To use the charts do the following: 1) find the constant-density c_f, c_{f0}, at the required R_θ from Fig. 1; 2) using this, find the adiabatic-wall c_f, $c_{f,aw}$ for the required M_e and R_θ from Fig. 2: only a rough interpolation for R_θ is needed to achieve 1% accuracy; 3) using c_f and assuming $T_{aw} = T_e(1+0.178M_e^2)$, find c_f for the required M_e and T_w from Fig. 3: again, the variation with R_θ is slow, and the variation with M_e is rapid only at low T_w/T_{aw} where it is unrealistic to require better than 5% accuracy even for comparative purposes.

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